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1987 J. Phys. A: Math. Gen. 20 6357

(http://iopscience.iop.org/0305-4470/20/18/035)

J. Phys. A: Math. Gen. 20 (1987) 6357-6370. Printed in the UK

Finite-size scaling for a relativistic Bose gas in an Einstein universe

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Received 27 May 1987

Abstract. The scaling hypothesis on the 'singular' part of the free-energy density of a finite system is examined in the context of a relativistic Bose gas confined to an Einstein universe of radius R. Finite-size effects in the various thermodynamic properties of the system, such as the free energy, the specific heat, the isothermal compressibility and the condensate density, are predicted in the regions of both first-order ($T < T_c$) and second-order ($T \simeq T_c$) phase transitions. To test these predictions, a detailed analytical study is carried out which includes the possibility of particle-antiparticle pair production in the system. The various predictions of the scaling hypothesis are fully borne out and the scaling functions governing the critical behaviour of the system are found to be universal, irrespective of the severity of the relativistic effects.

1. Introduction

Some time ago we carried out a detailed investigation of the critical behaviour of an ideal relativistic Bose gas confined to the background geometry of an Einstein universe (Singh and Pathria 1984, hereafter referred to as I). Taking into account the possibility of particle-antiparticle pair production in the system, we examined the twin problems of (i) the growth of the condensate fraction, ρ_0/ρ, and (ii) the behaviour of the specific heat at constant volume, c_v, as a function of temperature. Though many of the results obtained in that paper were valid over a wide range of temperatures—from $T \gtrsim T_c$ down to $T = 0$ K—our physical discussion was mostly centred at the region of the second-order phase transition ($T \simeq T_c$). It was remarkable that our findings on the finite-size effects in the relativistic case turned out to be qualitatively similar to the ones obtained by Al'taie (1978) in the corresponding non-relativistic case; quantitatively, of course, they depended significantly on the severity of the relativistic effects which, in turn, were determined by the parameter ρ/m^3, where ρ is (essentially) the 'number density' in the system while m is the particle mass.

Since then considerable progress has been made in understanding finite-size effects in systems undergoing phase transitions, on the basis of concepts such as scaling and hyperuniversality (Privman and Fisher 1984, Singh and Pathria 1985, 1986, Shapiro 1986) which enable us to make definitive predictions on the nature of these effects in the regions of both first-order ($T < T_c$) and second-order ($T \simeq T_c$) phase transitions (for a review of earlier developments and for a comprehensive bibliography on the subject, see Barber (1983)). Using these concepts we have recently explored the problem of a relativistic Bose gas in a Euclidean space of geometry $L^{d-d'} \times \infty^{d'}$ ($2 < d < 4$, $d' \leqslant 2$)

under periodic boundary conditions (Singh and Pathria 1987, hereafter referred to as II). On one hand, we made predictions, based on the scaling hypothesis on the 'singular' part of the free-energy density of the system, for quantities such as the specific heat, the isothermal compressibility and the condensate density which covered the temperature regime $T < T_c$ as well as $T = T_c$; on the other hand, we carried out an exact analysis of the problem to derive explicit expressions for these quantities from which desired finite-size effects could be extracted analytically. A comparison of the two sets of results showed that each and every prediction of the scaling hypothesis was, in fact, true. At that point we felt it natural to extend these considerations to the situation where the Euclidean space is replaced by an Einstein universe, with $d = 3$ and $d' = 0$, which is inherently finite (with volume $2\pi^2R^3$, R being the radius of curvature of the 3-space) and conforms to boundary conditions that are inherently periodic (with period $2\pi R$). The main purpose of such an investigation would be to see if the scaling hypothesis, originally formulated for conventional geometries, stayed valid for curved spaces as well. Intuitively we felt that, since the major thrust of the arguments employed in the formulation of the hypothesis and in the derivation of the predictions came from the critical behaviour of the bulk system, the resulting formulae should apply to all finite-sized systems, irrespective of the geometrical nature of the space available to them. As the results reported here will show, this expectation is indeed upheld.

In § 2 we introduce the scaling hypothesis for a system of non-interacting bosons confined to a 3-space of uniform radius of curvature R and obtain general expressions for the various physical quantities of interest. In § 3 we make specific predictions for these quantities in different regimes of temperature. In § 4 we derive the corresponding analytical results from which finite-size effects are readily obtained. A detailed comparison of the two sets of results is carried out in § 5 where the scaling functions governing the critical behaviour of the various quantities are written down explicitly. The agreement between the two sets of results is seen to be perfect, which validates the scaling hypothesis for systems in curved spaces as well. Finally, in § 6, we conclude the paper with some closing remarks on this problem.

2. Formulation of the problem

In accordance with II, we propose that the 'singular' part of the free-energy density of a system of non-interacting bosons confined to an Einstein universe of radius R may be expressed in the form

$$f^{(s)}(T, \nu_0; R) \approx TR^{-d}Y(x_1, x_2) \qquad d = 3 \tag{1}$$

where

$$\begin{aligned} x_1 &= \tilde{C}_1 \tilde{t} R^{1/\nu} \qquad & x_2 &= \tilde{C}_2(\nu_0/T)R^{\Delta/\nu} \\ \nu &= 1/(d-2) = 1 \qquad & \Delta/\nu &= \tfrac{1}{2}(d+2) = \tfrac{5}{2}. \end{aligned} \tag{2}$$

Here, ν_0 stands for the 'Bose field' (see Gunton and Buckingham 1968) conjugate to the 'order parameter' M_0 ($\equiv \rho_0^{1/2}$, ρ_0 being the condensate density in the system), $\tilde{t}$ is the generalised temperature variable, while other quantities have their usual meanings. In particular, $\tilde{C}_1$ and $\tilde{C}_2$ are certain non-universal, system-dependent scale factors whose precise form can be determined from a knowledge of the thermodynamic behaviour of the corresponding bulk system. The function $Y(x_1, x_2)$ is then a universal function, common to all systems in the same universality class as the given system.

According to (1), the 'singular' part of the zero-field specific heat per unit volume of the system will be given by

$$c^{(s)}(T, 0; R) = -T\left(\frac{\partial^2 f^{(s)}}{\partial T^2}\right)_{\rho, \nu_0=0} \approx \left(T\frac{\partial(\tilde{C}_1\tilde{t})}{\partial T}\right)^2_\rho R^{-1} Y_{(1)}(x_1) \tag{3}$$

where $Y_{(1)}(x_1) = -(\partial^2 Y/\partial x_1^2)_{x_2=0}$. The isothermal compressibility, on the other hand, will be given by (see appendix A of II)

$$\kappa(T, 0; R) = \left(\rho^2 \frac{\partial^2 f^{(s)}}{\partial \rho^2}\right)^{-1}_{T, \nu_0=0} \approx T^{-1}\left(\rho\frac{\partial(\tilde{C}_1\tilde{t})}{\partial \rho}\right)^{-2}_T R Y_{(2)}(x_1) \tag{4}$$

where ρ is the overall 'charge density' in the system, as defined in equation (61), while $Y_{(2)}(x_1) = -1/Y_{(1)}(x_1)$. In view of the straightforward relationship between the scaling functions $Y_{(1)}$ and $Y_{(2)}$, we conclude that, in all regimes of temperature, the product $c^{(s)}\kappa$ is independent of the size of the system,

$$c^{(s)}\kappa \approx -T[\rho^{-1}(\partial\rho/\partial T)_{\tilde{C}_1\tilde{t}}]^2 \tag{5}$$

and hence may be obtained directly from the bulk behaviour. For simplicity, therefore, we may in the following concentrate on only one of these quantities and refer to the other only when need arises. As regards condensate density in the system, one can argue that

$$\rho_0(T, 0; R) \equiv |M(T, 0; R)|^2 \approx \tilde{C}_2^2 R^{-1} P(x_1) \tag{6}$$

where $P(x_1)$ is the corresponding scaling function.

As shown in II, the non-universal parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ appearing in the above expressions can be expressed in terms of quantities pertaining to the bulk system. In the case under study, they turn out to be

$$\tilde{C}_1\tilde{t} = m^2\beta[W(\beta, m) - W(\beta_c, m)] \qquad \tilde{C}_2 = (m/\beta)^{1/2} \tag{7}$$

where

$$W(\beta, m) = 2\sum_{j=1}^{\infty} (j\beta m)^{-1}\sinh(j\beta m)K_2(j\beta m) \qquad \beta = 1/T \tag{8}$$

$K_\nu(z)$ being the modified Bessel functions while the bulk critical point, $\beta = \beta_c$, is determined by the relationship

$$W(\beta_c, m) = 2\pi^2\rho/m^3. \tag{9}$$

It may as well be noted that the bulk condensate density $\rho_0(T, 0; \infty)$ in this case is given by (Singh and Pandita 1983)

$$\rho_0(T, 0; \infty) = \begin{cases} 0 & \beta \leqslant \beta_c \qquad (10a) \\ \rho[1 - W(\beta, m)/W(\beta_c, m)] & \beta \geqslant \beta_c \qquad (10b) \end{cases}$$

so that

$$[\tilde{C}_1|\tilde{t}|]_{T<T_c} = (2\pi^2\beta/m)\rho_0(T, 0; \infty). \tag{11}$$

It follows that

$$\tilde{C}_1|\tilde{t}| \to \begin{cases} 2\pi^2\beta\rho/m \sim T^{-1} & T \to 0 \qquad (12a) \\ C_1|t| & T \to T_c \qquad (12b) \end{cases}$$

where

$$C_1 = m^2\beta_c^2|\mathrm{d}W/\mathrm{d}\beta|_c \tag{13}$$

while t is the conventional temperature variable

$$t = (T - T_c)/T_c = (\beta_c - \beta)/\beta. \tag{14}$$

Finally, we note that equation (5) in this case assumes the explicit form

$$c^{(s)}\kappa = -\beta[W(\beta, m) - W(\beta_c, m) + \beta(\mathrm{d}W/\mathrm{d}\beta)]^2/W^2(\beta_c, m) \tag{15}$$

with the limiting results

$$c^{(s)}\kappa \simeq \begin{cases} -\beta & \beta \to \infty \quad (16a) \\ -\beta_c^3(\mathrm{d}W/\mathrm{d}\beta)_c^2/W^2(\beta_c, m) & \beta \simeq \beta_c. \quad (16b) \end{cases}$$

We are now in a position to make predictions about the various physical quantities and the various mathematical functions involved.

3. Prediction of finite-size effects

(a) For $T \gtrsim T_c$ and $R \to \infty$, we expect our hypothesis to reproduce the standard bulk result for $c^{(s)}$, namely

$$c^{(s)} \approx -E_+ t \qquad T \gtrsim T_c,\ R \to \infty \tag{17}$$

where E_+ is non-universal. To recover (17) from (3), we require that, as $x_1 \to +\infty$, the scaling function $Y_{(1)}(x_1)$ assume the asymptotic form

$$Y_{(1)}(x_1) \approx -Y_+ x_1 \qquad x_1 \to +\infty \tag{18}$$

with Y_+ universal and such that

$$E_+ = Y_+ C_1^3 \tag{19}$$

where C_1 is given by equation (13). The corresponding results for $f^{(s)}$ would be

$$f^{(s)} \approx F_+ t^3 \qquad Y(x_1) \approx Z_+ x_1^3 \tag{20}$$

with

$$F_+ = Z_+ T_c C_1^3. \tag{21}$$

Now, since F_+ would also be equal to $\frac{1}{6}T_c E_+$, it follows that

$$Z_+ = \tfrac{1}{6} Y_+. \tag{22}$$

As regards condensate density, we note that in this regime the total number of particles in the ground state is expected to be O(1); accordingly, $\rho_0(T; R) = \mathrm{O}(R^{-3})$. This requires that

$$P(x_1) \approx P_+ x_1^{-2} \qquad x_1 \to +\infty \tag{23}$$

with P_+ universal. It follows that

$$\rho_0(T; R) \approx P_+ C_1^{-2} C_2^3 t^{-2} R^{-3} \qquad T \gtrsim T_c,\ R \to \infty \tag{24}$$

where

$$C_2 = (m/\beta_c)^{1/2}. \tag{25}$$

(b) For $T < T_c$ and $R \to \infty$, the variable $x_1 \to -\infty$. The scaling function $Y_{(1)}(x_1)$ may then assume the form

$$Y_{(1)}(x_1) \approx -Y_-|x_1|^{-\sigma} \qquad x_1 \to -\infty \tag{26}$$

with Y_- universal and σ as yet unknown. The resulting expressions for $c^{(s)}$ and κ would be

$$c^{(s)} \approx -\frac{Y_-(m^2\beta)^{2-\sigma}[W(\beta, m) - W(\beta_c, m) + \beta(\mathrm{d}W/\mathrm{d}\beta)]^2}{[W(\beta_c, m) - W(\beta, m)]^{\sigma} R^{1+\sigma}} \tag{27}$$

and

$$\kappa \approx \frac{m^{2(\sigma-2)}[W(\beta_c, m) - W(\beta, m)]^{\sigma} R^{1+\sigma}}{Y_-\beta^{1-\sigma} W^2(\beta_c, m)}. \tag{28}$$

As $T \to 0$, $W(\beta, m)$ vanishes as $T^{3/2}$; equations (27) and (28) then reduce to

$$c^{(s)} \approx -Y_-[m^2\beta W(\beta_c, m)]^{2-\sigma} R^{-(1+\sigma)} \tag{29}$$

and

$$\kappa \approx Y_-^{-1}[m^2 W(\beta_c, m)]^{\sigma-2}\beta^{\sigma-1} R^{1+\sigma}. \tag{30}$$

Now, from general considerations based on the relationship between (i) the isothermal compressibility of a fluid on one hand and (ii) the integral of its correlation function over the space occupied by the system on the other (Pathria 1972), we find that, in the limit $T \to 0$,

$$\kappa \approx V/T = 2\pi^2\beta R^3. \tag{31}$$

Comparing (30) and (31), we readily infer that

$$\sigma = 2 \qquad Y_- = 1/2\pi^2. \tag{32}$$

The limiting form of the scaling function $Y_{(1)}(x_1)$, as $x_1 \to -\infty$, is thus completely determined:

$$Y_{(1)}(x_1) \approx -1/2\pi^2|x_1|^2 \qquad x_1 \to -\infty. \tag{33}$$

Consequently, the quantities $c^{(s)}$ and κ, for all $T < T_c$, are given explicitly by the expressions

$$c^{(s)} \approx -\frac{1}{(2\pi^2 R^3)}\left(\frac{W(\beta, m) - W(\beta_c, m) + \beta(\mathrm{d}W/\mathrm{d}\beta)}{W(\beta_c, m) - W(\beta, m)}\right)^2 \tag{34}$$

and

$$\kappa \approx (2\pi^2 R^3)\beta\left(\frac{W(\beta_c, m) - W(\beta, m)}{W(\beta_c, m)}\right)^2 = \frac{V}{T}\left(\frac{\rho_0}{\rho}\right)^2 \tag{35}$$

respectively. Equation (35) brings out a remarkable (and somewhat unsuspected) relationship between the isothermal compressibility of the finite-size system under study and the condensate density in the corresponding bulk system. Finally, as $T \to T_{c_-}$, equations (34) and (35) assume the form

$$c^{(s)} \approx -\frac{1}{2\pi^2 R^3|t|^2} \qquad \kappa = \frac{2\pi^2\beta_c^3(\mathrm{d}W/\mathrm{d}\beta)_c^2 R^3|t|^2}{W^2(\beta_c, m)}. \tag{36}$$

The asymptotic behaviour of the scaling function, $Y(x_1)$, for the free-energy density of the system may, in this regime, be derived from equation (33). One obtains

$$Y(x_1) \approx Z_-|x_1| - \frac{1}{2\pi^2}(\ln|x_1| + \text{constant}) \qquad x_1 \to -\infty \tag{37}$$

with Z_- universal. The corresponding expression for $f^{(s)}$ would be

$$f^{(s)} \approx \frac{Z_-\tilde{C}_1|\tilde{t}|T}{R^2} - \frac{T}{2\pi^2 R^3}[\ln(\tilde{C}_1|\tilde{t}|R) + \text{constant}] \qquad T < T_c, R \to \infty. \tag{38}$$

As $T \to 0$, the foregoing expression approaches the limiting value (see equation (12*a*))

$$(f^{(s)})_0 = Z_-(2\pi^2\rho/mR^2). \tag{39}$$

Now, on physical grounds we expect that the free-energy density in this limit would be the same as the ground-state energy density of the system, i.e. $\rho/2mR^2$ (see equation (52), with $n = 1$). We therefore conclude that

$$Z_- = 1/4\pi^2. \tag{40}$$

As regards condensate density, we first of all expect to recover the bulk result, namely

$$\rho_0 \approx B^2|t| \qquad T \leqslant T_c, R \to \infty \tag{41}$$

where B^2 is non-universal. This requires that, for $x_1 \to -\infty$, the scaling function $P(x_1)$ of equation (6) be of the form

$$P(x_1) \approx P_-|x_1| \qquad x_1 \to -\infty \tag{42}$$

with P_- universal and such that

$$B^2 = P_-C_1C_2^2. \tag{43}$$

It follows that, for all $T < T_c$,

$$\rho_0(T; \infty) = P_-\tilde{C}_1\tilde{C}_2^2|\tilde{t}| = P_-m^3[W(\beta_c, m) - W(\beta, m)]. \tag{44}$$

To recover the known result for $\rho_0(T; \infty)$ (see equations (9) and (10*b*)) we require that

$$P_- = 1/2\pi^2. \tag{45}$$

For the finite-size effect in ρ_0, we follow the line of argument presented in II and conclude that

$$\rho_0(T; R) - \rho_0(T; \infty) \approx Q_-mT/R \qquad T < T_c, R \to \infty \tag{46}$$

with Q_- universal.

(c) In the 'core' region, where $|x_1| = O(1)$ and hence $|t| = O(R^{-1})$, the functions $f^{(s)}$, $c^{(s)}$, κ and ρ_0, for a fixed value of x_1, are proportional to R^{-3}, R^{-1}, R and R^{-1}, respectively (see equations (1), (3), (4) and (6)). Accordingly, the quantities

$$f^{(s)}(T_c; R)R^3T_c^{-1} = U_f \tag{47}$$

$$c^{(s)}(T_c; R)RC_1^{-2} = U_c \tag{48}$$

$$\kappa(T_c; R)R^{-1}T_c^{-1}[\rho(\partial T_c/\partial\rho)]^2C_1^2 = U_\kappa \tag{49}$$

and

$$\rho_0(T_c; R)RC_2^{-2} = U_\rho \tag{50}$$

evaluated at the erstwhile critical point $T = T_c$, should be universal.

This completes the set of predictions, following from hypothesis (1), which will now be tested in the case of an ideal, relativistic Bose gas confined to an Einstein universe of radius R.

4. Thermodynamics of an ideal relativistic Bose gas in an Einstein universe

We consider an ideal Bose gas composed of N_1 particles and N_2 antiparticles, each of mass m, confined to an Einstein universe of (spatial) radius R. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number Q $(=N_1-N_2)$, rather than of N_1 and N_2 separately; the conserved quantity Q may be looked upon as a kind of generalised 'charge'. In equilibrium the chemical potentials of the two species will be equal and opposite: $\mu_1=-\mu_2=\mu$, say, with the result that (Haber and Weldon 1981, 1982)

$$N_1=\sum_\varepsilon\{\exp[\beta(\varepsilon-\mu)]-1\}^{-1} \qquad N_2=\sum_\varepsilon\{\exp[\beta(\varepsilon+\mu)]-1\}^{-1} \tag{51}$$

where $\varepsilon=(k^2+m^2)^{1/2}$; for economy, we employ units such that $\hbar=c=k_B=1$. Note that both ε and μ include the rest energy m of the particle (or the antiparticle) and, for the mean occupation numbers in the various states to be positive definite, we must have $|\mu|\leqslant\varepsilon_{\min}$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of Q, μ then stays positive under all circumstances. Without loss of generality, we shall assume that this indeed is the case.

The eigenvalues, k_n, of the wavenumber k for a free particle confined to the Einstein universe are given by (see, for example, Schrödinger 1938)

$$k_n=n/R \qquad n=1,2,3,\ldots \tag{52}$$

with multiplicity $g_n=n^2$. The pressure $\mathscr{P}$ of the system is then given by

$$\begin{aligned}\mathscr{P}&=-\frac{1}{\beta V}\sum_{n=1}^{\infty}n^2(\ln\{1-\exp[-\beta(\varepsilon_n-\mu)]\}+\ln\{1-\exp[-\beta(\varepsilon_n+\mu)]\})\\&=\frac{2}{\beta V}\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j}\sum_{n=1}^{\infty}n^2\exp\left[-j\beta m\left(1+\frac{n^2}{m^2R^2}\right)^{1/2}\right].\end{aligned} \tag{53}$$

Following the procedure outlined in I, equation (53) can be rendered into the form

$$\mathscr{P}=\frac{2m^2R^3}{\beta^2 V}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\sum_{q=-\infty}^{\infty}\left(\frac{K_2(\beta mz)}{z^2}-(\beta mq'^2)\frac{K_3(\beta mz)}{z^3}\right) \tag{54}$$

where

$$z=(j^2+q'^2)^{1/2} \qquad q'=(2\pi R/\beta)q. \tag{55}$$

The term with $q=0$ yields the bulk result,

$$\mathscr{P}_B(\beta,\mu)=\frac{m^4}{\pi^2}\sum_{j=1}^{\infty}(j\beta m)^{-2}\cosh(j\beta\mu)K_2(j\beta m) \tag{56}$$

while terms with $q\neq 0$ represent finite-size effects in $\mathscr{P}$. In the case of the latter, the summation over j may be replaced by an integration, which entails errors at most $O(e^{-R/\lambda})$ where λ denotes the mean thermal wavelength $(2\pi\beta/m)^{1/2}$ (or the Compton wavelength $1/m$) of the particles. Thus, correct to all powers of λ/R,

$$\mathscr{P}=\mathscr{P}_B(\beta,\mu)-\frac{y^3}{2\pi^{9/2}\beta R^3}[\mathscr{K}_{3/2}(y)+\mathscr{K}_{1/2}(y)] \tag{57}$$

where y is the thermogeometric parameter of the problem,

$$y=\pi(m^2-\mu^2)^{1/2}R \tag{58}$$

while $\mathcal{K}_\nu(y)$ are special cases of the functions $\mathcal{K}(\nu|l; y)$ introduced earlier (see Singh and Pathria 1985, 1987), namely

$$\mathcal{K}(\nu|l; y) = \sum_{q(l)}{}' \frac{K_\nu(2yq)}{(yq)^\nu} \qquad q = (q_1^2 + \ldots + q_l^2)^{1/2} > 0 \tag{59}$$

with $l = 1$. Accordingly,

$$\mathcal{K}_\nu(y) = \sum_{q=-\infty}^{\infty}{}' \frac{K_\nu(2y|q|)}{(y|q|)^\nu} = 2\sum_{q=1}^{\infty} \frac{K_\nu(2yq)}{(yq)^\nu}. \tag{60}$$

The corresponding expression for the 'charge density' ρ can be obtained by noting that

$$\rho = (N_1 - N_2)/V = (\partial \mathcal{P}/\partial\mu)_\beta \tag{61}$$

with the result that

$$\rho = \rho_{\mathrm{B}}(\beta, \mu) - \frac{\mu y}{\pi^{5/2}\beta R}\mathcal{K}_{-1/2}(y) \tag{62}$$

where

$$\rho_{\mathrm{B}}(\beta, \mu) = \frac{m^3}{\pi^2}\sum_{j=1}^{\infty} (j\beta m)^{-1} \sinh(j\beta\mu) K_2(j\beta m). \tag{63}$$

It may be mentioned here that in arriving at equation (62) we have also made use of the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}y}[y^{2\nu}\mathcal{K}_\nu(y)] = -2y^{2\nu-1}\mathcal{K}_{\nu-1}(y). \tag{64}$$

In the region of phase transition ($\mu \simeq m$), equations (57) and (62) take the form

$$\mathcal{P} = \mathcal{P}_{\mathrm{B}}(\beta, m) + (\mu - m)\rho_{\mathrm{B}}(\beta, m) + \frac{(m^2-\mu^2)^{3/2}}{6\pi\beta} - \frac{y^3}{2\pi^{9/2}\beta R^3} \times [\mathcal{K}_{3/2}(y) + \mathcal{K}_{1/2}(y)] + \mathrm{O}(y^4/R^4) \tag{65}$$

and

$$\rho = \rho_{\mathrm{B}}(\beta, m) - \frac{m(m^2-\mu^2)^{1/2}}{2\pi\beta} - \frac{my}{\pi^{5/2}\beta R}\mathcal{K}_{-1/2}(y) + \mathrm{O}(y^2/R^2). \tag{66}$$

The thermal free-energy density of the system is then given by

$$f = \frac{F - mQ}{V} = \frac{(\mu Q - \mathcal{P}V) - mQ}{V} = (\mu - m)\rho - \mathcal{P} \tag{67}$$

from which the 'singular' part of $f^{(s)}$ can be readily extracted. To the desired order in (y/R), we obtain

$$f^{(s)} = \frac{y^3}{12\pi^4\beta R^3}\left(1 + \frac{6}{\sqrt{\pi}}[\mathcal{K}_{3/2}(y) + \mathcal{K}_{1/2}(y) + \mathcal{K}_{-1/2}(y)]\right). \tag{68}$$

In view of the fact that

$$\mathcal{K}_{-1/2}(y) = \sqrt{\pi}/(\mathrm{e}^{2y} - 1) \tag{69}$$

equation (66), again to the desired order in (y/R), gives

$$\rho = \rho_{\mathrm{B}}(\beta, m) - \frac{m}{2\pi^2\beta R} y \coth y. \tag{70}$$

Now, the bulk critical point, $\beta = \beta_c$, is determined by the condition $\rho_B(\beta_c, m) = \rho$ which, according to (63), may be written as

$$\sum_{j=1}^{\infty} (j\beta_c m)^{-1} \sinh(j\beta_c m) K_2(j\beta_c m) = \pi^2 \rho / m^3. \tag{71}$$

In terms of the function $W(\beta, m)$ defined in (8), equations (70) and (71) take the form

$$m^2 \beta R[W(\beta, m) - W(\beta_c, m)] = y \coth y \tag{72}$$

and

$$W(\beta_c, m) = 2\pi^2 \rho / m^3 \tag{73}$$

respectively. Equations (68) and (72) constitute the central results of this calculation.

Corresponding results for the 'singular' part of the specific heat and the isothermal compressibility of the system turn out to be

$$\begin{aligned} c^{(s)} &= -\beta^2 [\partial^2(\beta f^{(s)})/\partial\beta^2]_\rho \\ &= -\frac{m^4 \beta^2 [W(\beta, m) - W(\beta_c, m) + \beta(\mathrm{d}W/\mathrm{d}\beta)]^2 y}{2\pi^4 R\{1 + (4/\sqrt{\pi})[\mathcal{H}_{-1/2}(y) - \mathcal{H}_{-3/2}(y)]\}} \end{aligned} \tag{74}$$

and

$$\begin{aligned} \kappa &= \rho^{-2} (\partial\rho/\partial\mu)_\beta \\ &= \frac{m^2 R\{1 + (4/\sqrt{\pi})[\mathcal{H}_{-1/2}(y) - \mathcal{H}_{-3/2}(y)]\}}{2\beta\rho^2 y} \end{aligned} \tag{75}$$

respectively. It will be noted that the product $c^{(s)}\kappa$ is indeed independent of R,

$$c^{(s)}\kappa = -(m^6 \beta / 4\pi^4 \rho^2)[W(\beta, m) - W(\beta_c, m) + \beta(\mathrm{d}W/\mathrm{d}\beta)]^2 \tag{76}$$

and agrees with the bulk result (15). As regards condensate density, we simply quote the expression derived in I, namely

$$\rho_0 \simeq m / \beta R(y^2 + \pi^2) \tag{77a}$$

$$= \rho\left(1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right) + \frac{m}{\pi^2 \beta R}\left(\frac{3}{2} + y^2 \sum_{q=2}^{\infty} \frac{1}{y^2 + \pi^2 q^2}\right). \tag{77b}$$

Recalling the bulk result (10*b*), we find that the finite-size effect in ρ_0, for all $T \leqslant T_c$, is given by

$$\rho_0(T; R) - \rho_0(T; \infty) = \frac{m}{\pi^2 \beta R}\left(\frac{3}{2} + y^2 \sum_{q=2}^{\infty} \frac{1}{y^2 + \pi^2 q^2}\right). \tag{78}$$

We shall now compare our analytical results for the various quantities of interest with the predictions made in § 3.

5. Verification of the scaling predictions

We start with the observation that, with $\tilde{C}_1 \tilde{t}$ given by (7), the constraint equation (72) assumes the remarkably simple form

$$x_1 = y \coth y \tag{79}$$

which determines y as a function of the scaled variable x_1. It is now straightforward to see that expressions (68), (74), (75) and (77) indeed conform to the scaled forms (1), (3), (4) and (6), respectively, with scaling functions

$$Y(x_1)=\frac{y^3}{12\pi^4}\{1+(6/\sqrt{\pi})[\mathscr{K}_{3/2}(y)+\mathscr{K}_{1/2}(y)+\mathscr{K}_{-1/2}(y)]\} \tag{80}$$

$$Y_{(1)}(x_1)=-[Y_{(2)}(x_1)]^{-1}=-\frac{y}{2\pi^4}\left(1+\frac{4}{\sqrt{\pi}}[\mathscr{K}_{-1/2}(y)-\mathscr{K}_{-3/2}(y)]\right)^{-1} \tag{81}$$

and

$$P(x_1)=(y^2+\pi^2)^{-1}. \tag{82}$$

Of course, to express these functions in terms of the variable x_1, we have to eliminate y with the help of equation (79). For this, we consider the various regimes of temperature one by one.

(*a*) $T \gtrsim T_c$, $R\to\infty$. In this regime $x_1\to+\infty$, with the result that y diverges as

$$y(x_1)\approx x_1[1-2\exp(-2x_1)] \qquad x_1\to+\infty. \tag{83}$$

In view of the fact that, for $y\gg 1$,

$$\mathscr{K}_\nu(y)\approx\sqrt{\pi}\,y^{-(\nu+1/2)}\,\mathrm{e}^{-2y} \tag{84}$$

scaling functions (80) and (81) assume the form

$$Y(x_1)\approx\frac{x_1^3}{12\pi^4}\left(1+\frac{6}{x_1}\exp(-2x_1)\right) \tag{85}$$

and

$$Y_{(1)}(x_1)=-[Y_{(2)}(x_1)]^{-1}\approx-\frac{x_1}{2\pi^4}[1+4x_1\exp(-2x_1)]. \tag{86}$$

Equations (85) and (86) agree with predictions (18) and (20), respectively, with

$$Z_+=1/12\pi^4 \qquad Y_+=1/2\pi^4 \tag{87}$$

in further agreement with prediction (22). At the same time, corrections to standard bulk behaviour turn out to be exponentially small. It seems instructive to express these corrections in terms of the variable ξ/R, where ξ is the 'correlation length' appropriate to the case under study (see Singh and Pandita 1983), namely

$$\xi=(m^2-\mu^2)^{-1/2}=\pi R/y\simeq\pi R/x_1=\pi/C_1 t. \tag{88}$$

We thus get

$$f^{(s)}\approx\frac{C_1^3t^3}{12\pi^4\beta_c}\left(1+\frac{6\xi}{\pi R}\exp(-2\pi R/\xi)\right) \tag{89}$$

$$c^{(s)}\approx-\frac{C_1^3t}{2\pi^4}\left(1+\frac{4\pi R}{\xi}\exp(-2\pi R/\xi)\right) \tag{90}$$

and

$$\kappa\approx\frac{m^2}{2\rho^2\beta_c C_1 t}\left(1-\frac{4\pi R}{\xi}\exp(-2\pi R/\xi)\right). \tag{91}$$

It is gratifying that the exponent $(2\pi R/\xi)$ appearing here is precisely what one would expect on the basis of the corresponding flat-space results (Singh and Pathria 1987), for $2\pi R$ is exactly the 'period' in space for the wavefunctions appropriate to the Einstein universe—just as L was in the case of the flat space.

Finally, the scaling function for the condensate density in this regime takes the form

$$P(x_1) \approx x_1^{-2} \qquad x_1 \to +\infty \tag{92}$$

which agrees with prediction (23), with $P_+ = 1$. At the same time, equation (77a) gives

$$\rho_0 \approx m/\beta_c C_1^2 t^2 R^3 \tag{93}$$

which agrees with equations (24) and (25).

(b) $T < T_c$ and $R \to \infty$. In this regime $x_1 \to -\infty$ and by equation (79), which may be written as

$$x_1 = y^2 \sum_{q=-\infty}^{\infty} \frac{1}{y^2 + \pi^2 q^2} \tag{94}$$

$y^2 \to -\pi^2$. This is not surprising because, in this limit, the chemical potential μ tends to the lowest eigenvalue of energy, i.e. $\mu \to \varepsilon_1 = (m^2 + 1/R^2)^{1/2} \simeq m + 1/2mR^2$, with the result that $y^2 = \pi^2 R^2 (m^2 - \mu^2) \to -\pi^2$. More accurately,

$$y^2 \approx -\pi^2 + 2\pi^2 |x_1|^{-1} - 3\pi^2 |x_1|^{-2} \qquad x_1 \to -\infty. \tag{95}$$

To determine the behaviour of the various scaling functions in this regime we must first of all express them as explicit functions of the variable y^2, so that a smooth passage from $y^2 > 0$ to $y^2 < 0$ may be made without encountering any 'awkwardness' at $y^2 = 0$. This has been done in the appendix, which leads to the following results.

As $y^2 \to -\pi^2$, the scaling function $Y_{(1)}(x_1)$ behaves asymptotically as (see equations (95) and (A5))

$$Y_{(1)}(x_1) \approx -\frac{(y^2 + \pi^2)^2}{8\pi^6} \approx -\frac{1}{2\pi^2 |x_1|^2} \tag{96}$$

in perfect agreement with prediction (33). It follows that this will yield the same expressions for $c^{(s)}$ and κ as shown in equations (34)–(36). As for the function $Y(x_1)$, we obtain, with the help of equations (95) and (A9),

$$\begin{aligned} Y(x_1) &\approx \frac{1}{2\pi^2}\left[\frac{\pi^2}{y^2 + \pi^2} + \ln\left(1 + \frac{y^2}{\pi^2}\right) + \text{constant}\right] \\ &\approx \frac{1}{4\pi^2}|x_1| - \frac{1}{2\pi^2}\ln|x_1| + \text{constant} \end{aligned} \tag{97}$$

in perfect agreement with predictions (37) and (40); this will obviously yield the same expression for $f^{(s)}$ as shown in equation (38). As regards the constant term in (97), it could not be predicted on the basis of general considerations of § 3. It can, however, be determined numerically with the help of equations (94) and (A9).

Finally, the scaling function for ρ_0 turns out to be (see equations (82) and (95))

$$P(x_1) = \frac{1}{y^2 + \pi^2} \approx \frac{1}{2\pi^2}|x_1| + \frac{3}{4\pi^2}. \tag{98}$$

The leading term in (98), which yields the bulk result for ρ_0, indeed agrees with predictions (42) and (45); the next term yields the finite-size effect, $\rho_0(T; R) - \rho_0(T; \infty)$, which agrees with prediction (46), with $Q_- = 3/4\pi^2$.

(*c*) In the 'core' region, where $|x_1| = O(1)$ and hence $|t| = O(R^{-1})$, y^2 would be significantly different from $-\pi^2$ but $|y^2|$ would still be of order unity. Accordingly, $f^{(s)}$, $c^{(s)}$, κ and ρ_0 would be of order R^{-3}, R^{-1}, R and R^{-1}, respectively (see equations (68), (74), (75) and (77*a*)). This is indeed the way it was predicted in § 3. More specifically, however, we should check that the quantities (47)-(50), evaluated at the erstwhile critical point, $T = T_c$, are indeed universal.

For this, we note that, by equation (79), $(y^2)_c = -\pi^2/4$. It then follows (see equations (68), (80) and (A9)) that

$$U_f = Y(0) = \frac{1}{4\pi^4}\left(\zeta(3) + \tfrac{1}{4}\pi^2 + 2\pi^2 \sum_{q=1}^{\infty} [q^2 \ln(1 - 1/4q^2) + \tfrac{1}{4}]\right) \tag{99}$$

which is clearly a universal number. Similarly,

$$U_c = Y_{(1)}(0) = -\left(8\pi^2 \sum_{q=1}^{\infty} \frac{q^2}{(q^2 - \frac{1}{4})^2}\right)^{-1} = -\frac{1}{2\pi^4} \tag{100}$$

another universal number. Next, we observe that the derivative $(\partial\beta_c/\partial\rho)$ for the bulk system is given by (see equation (73))

$$(\partial\beta_c/\partial\rho) = W_c/\rho(\partial W/\partial\beta)_c \tag{101}$$

with the result that

$$U_\kappa = Y_{(2)}(0) = 2\pi^4. \tag{102}$$

Finally, by equations (50) and (77), we obtain the universal number

$$U_\rho = 4/3\pi^2. \tag{103}$$

This completes the verification of the various predictions made in § 3.

6. Concluding remarks

In this paper we have shown analytically that the various predictions of the finite-size scaling hypothesis are fully borne out in the case of an ideal relativistic Bose gas confined to an Einstein universe of radius R. With pair production included, the scaling functions governing the thermodynamic behaviour of the system, for $T < T_c$ as well as $T = T_c$, are found to be universal, irrespective of the severity of the relativistic effects. The influence of the latter enters only through non-universal, model-dependent (and, in general, temperature-dependent) parameters $\tilde{C}_1 \tilde{t}$ and $\tilde{C}_2$ which are completely determined from the properties of the corresponding bulk system. From a cosmological point of view, these calculations may find relevance in the context of a primordial massive photon gas, as discussed by Kuzmin and Shaposhnikov (1979), with Bose-Einstein condensation playing a vital role in the early epoch of the universe.

A natural generalisation of the present study would be to investigate the behaviour of an ideal relativistic Bose gas confined to the space $\mathcal{S}^{d-d'} \times \mathcal{R}^{d'}$ which consists of an infinite Euclidean space of dimensionality d' associated with a finite curved space of dimensionality $d - d'$. A case of immediate interest would be the one for which $d = 3 + d'$, with $0 \leq d' < 1$; the limit $d' \to 1$ may then elucidate the influence of the 'curvature of the $\mathcal{S}$ space' on the passage of the system from the hyperscaling regime ($d < 4$) into the mean-field regime ($d > 4$) through the marginal dimensionality $d = 4$. Such a study would supplement a recent investigation by Cardy (1985) who has examined the behaviour of the spherical model of ferromagnetism in the space $\mathcal{S}^{d-1} \times \mathcal{R}^1$, with $2 < d < 4$. Work along these lines is in progress.

Acknowledgment

Financial support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

Appendix

To determine the behaviour of the scaling functions $Y(x_1)$ and $Y_{(1)}(x_1)$ for general y, we start with the function

$$\mathcal{K}_{-1/2}(y) = \sqrt{\pi} \sum_{q=1}^{\infty} e^{-2yq} = \tfrac{1}{2}\sqrt{\pi}(\coth y - 1) \tag{A1}$$

and make use of the identity (see Morse and Feshbach 1953)

$$\sum_{q=-\infty}^{\infty} e^{-2y|q|} = \coth y = y \sum_{q=-\infty}^{\infty} \frac{1}{y^2 + \pi^2 q^2} \tag{A2}$$

to obtain

$$y^{-1}\mathcal{K}_{-1/2}(y) = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{y} + \sum_{q=-\infty}^{\infty} \frac{1}{y^2 + \pi^2 q^2}\right). \tag{A3}$$

With the help of the recurrence relation (64), this gives

$$y^{-3}\mathcal{K}_{-3/2}(y) = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{2y^3} + \sum_{q=-\infty}^{\infty} \frac{1}{(y^2 + \pi^2 q^2)^2}\right). \tag{A4}$$

The scaling function $Y_{(1)}(x_1)$ (see equation (81)) then takes the simple form

$$Y_{(1)}(x_1) = -\left(8\pi^6 \sum_{q=1}^{\infty} \frac{q^2}{(y^2 + \pi^2 q^2)^2}\right)^{-1}. \tag{A5}$$

Next, we observe that

$$\mathcal{K}_{1/2}(y) = \frac{\sqrt{\pi}}{y} \sum_{q=1}^{\infty} \frac{e^{-2yq}}{q} = -\frac{\sqrt{\pi}}{y} \ln(1 - e^{-2y}) \tag{A6}$$

so that

$$y\mathcal{K}_{1/2}(y) = \sqrt{\pi}[y - \ln(2\sinh y)] = \sqrt{\pi}\left[y - \ln(2y) - \sum_{q=1}^{\infty} \ln\left(1 + \frac{y^2}{\pi^2 q^2}\right)\right]. \tag{A7}$$

Again, using the recurrence relation (64), we obtain

$$y^3\mathcal{K}_{3/2}(y) = \frac{\sqrt{\pi}}{2}\left\{\zeta(3) - y^2 + 2y^2\ln(2y) - \tfrac{4}{3}y^3 + 2\sum_{q=1}^{\infty}\left[(y^2 + \pi^2 q^2)\ln\left(1 + \frac{y^2}{\pi^2 q^2}\right) - y^2\right]\right\}. \tag{A8}$$

Note that the constant of integration in this expression has been determined by using the actual value of the function at $y^2 = 0$. The scaling function $Y(x_1)$ (see equation (80)) is now obtained by combining equations (A3), (A7) and (A8), with the result that

$$Y(x_1) = \frac{1}{4\pi^4}\left\{\zeta(3) + 2\pi^2 \sum_{q=1}^{\infty} q^2\left[\ln\left(1 + \frac{y^2}{\pi^2 q^2}\right) - \frac{y^2}{y^2 + \pi^2 q^2}\right]\right\}. \tag{A9}$$

It seems important to point out here that, whereas expressions (A3), (A4), (A7) and (A8) for the functions $\mathscr{H}_\nu(y)$ contain terms (involving $\ln y$ and odd powers of y) which, as functions of the physical parameter $\mu(\simeq m - y^2/2\pi^2 mR^2)$, are singular at $\mu = m$, i.e. at $y^2 = 0$, no such terms appear in the final expressions, (A5) and (A9), for the scaling functions $Y(x_1)$ and $Y_{(1)}(x_1)$. Accordingly, the latter functions vary smoothly with μ (or, for that matter, with y^2) for all $\mu < \varepsilon_1$ (i.e. all $y^2 > -\pi^2$), as indeed should be the case for a finite system at $T > 0$ K.

References

Al'taie M B 1978 *J. Phys. A: Math. Gen.* **11** 1603
Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) pp 145–266
Cardy J L 1985 *J. Phys. A: Math. Gen.* **18** L757
Gunton J D and Buckingham M J 1968 *Phys. Rev.* **166** 152
Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497
—— 1982 *Phys. Rev.* D **25** 502
Kuzmin V A and Shaposhnikov M E 1979 *Phys. Lett.* **69A** 462
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill) p 467
Pathria R K 1972 *Statistical Mechanics* (Oxford: Pergamon) § 13.2
Privman V and Fisher M E 1984 *Phys. Rev.* B **30** 322
Schrödinger E 1938 *Commun. Pont. Acad. Sci.* **2** 321
Shapiro J 1986 *Phys. Rev. Lett.* **56** 2225
Singh S and Pandita P N 1983 *Phys. Rev.* A **28** 1752
Singh S and Pathria R K 1984 *J. Phys. A: Math. Gen.* **17** 2983
—— 1985 *Phys. Rev. Lett.* **55** 347
—— 1986 *Phys. Rev. Lett.* **56** 2226
—— 1987 *Phys. Rev.* A **35** 4814